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PERFECT SETS OF RANDOM REALS

BY

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ABSTRACT

We show that the existence of a perfect set of random reals over a model M of ZFC does not imply the existence of a dominating real over M, thus answering a well-known open question (see [BJ 1] and [JS 2]). We also prove that $\mathbb{B}\times\mathbb{B}$ (the product of two copies of the random algebra) neither adds a dominating real nor adds a perfect set of random reals (this answers a question that A. Miller asked during the logic year at MSRI).

Introduction

The goal of this work is to give several results concerning the relationship between perfect sets of random reals, dominating reals, and the product of two copies of the random algebra $\mathbb{B}$. Recall that $\mathbb{B}$ is the algebra of Borel sets of 2^ω modulo the null sets. Also, given two models $M \subseteq N$ of ZFC, we say that $g \in \omega^\omega \cap N$ is a **dominating real over** M iff $\forall f \in \omega^\omega \cap M\ \exists m \in \omega\ \forall n \geq m\ (g(n) > f(n))$; and $r \in 2^\omega \cap N$ is **random over** M iff r avoids all Borel null sets coded in M iff r is the real determined by some filter which is $\mathbb{B}$-generic over M (see [Je 1, section 42] for details).

A tree $T \subseteq 2^{<\omega}$ is **perfect** iff $\forall t \in T\ \exists s \supseteq t\ (s\hat{}\langle 0\rangle \in T\ \wedge s\hat{}\langle 1\rangle \in T)$. For a perfect tree T we let $[T] := \{f \in 2^\omega;\ \forall n\ (f\restriction n \in T)\}$ denote the set of its branches. Then $[T]$ is a perfect set (in the topology of 2^ω). Conversely, given a perfect set $S \subseteq 2^\omega$ there is perfect tree $T \subseteq 2^{<\omega}$ such that $[T] = S$. This allows

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us to confuse perfect sets and perfect trees in the sequel; in particular, we shall use the symbol T for both the tree and the set of its branches. — As a perfect tree is (essentially) a real, the statement *there is a perfect set of reals random over M in N* (where $M \subseteq N$ are again models of ZFC) asserts the existence of a certain kind of real in N over M; and thus we may ask how it is related to the existence of other kinds of reals (like dominating reals). This will be our main topic. — We recall that the existence of a random real does not imply the existence of a perfect set of random reals; in fact Cichoń showed that $\mathbb{B}$ does not add a perfect set of random reals [BJ 1, Theorem 2.1]. (Here, we say that a p.o. $\mathbb{P}$ *adds a perfect set of random reals* iff there is a perfect set of reals random over M in $M[G]$, where G is $\mathbb{P}$-generic over M; a similar definition applies to dominating reals etc.)

We note that being a perfect set of random reals over some model M of ZFC is absolute in the following sense: if $M \subseteq N_0 \subseteq N_1$ are models of ZFC, $T \in (2^{<\omega})^\omega \cap N_0$ is a perfect tree so that $[T] \cap N_0$ consists only of reals random over M, then every real in $[T] \cap N_1$ is random over M as well (see [Je 1, Lemma 42.3]).

We now state the main results of our work, and explain how they will be presented in §§ 1 – 3; then we will give some further motivation for the study of perfect sets of random reals, and close with some notation.

THE MAIN RESULTS AND THE ORGANIZATION OF THE PAPER. Using techniques of [Ba], Bartoszyński and Judah proved in [BJ 1, Theorem 2.7] that

(*) *given models of ZFC $M \subseteq N$ such that N contains a dominating real over M, $N[r]$ contains a perfect set of random reals over M, where r is random over N.*

Our first result shows that the converse does not hold (1.4 – 1.7).

THEOREM 1: *There is a p.o. which adds a perfect set of random reals and does not add dominating reals.*

The framework for proving Theorem 1 (developed in § 1) will enable us to give general preservation results for *not adding dominating reals* in both finite support iterations and finite support products of *ccc* forcing notions (1.8). The former will be exploited in 1.9 to discuss cardinal invariants closely related to our subject. As a special instance of the latter we shall show (1.10)

THEOREM 2: $\mathbb{B} \times \mathbb{B}$ *does not add dominating reals.*

(This result was proved earlier by Shelah but never published.) The algebra $\mathbb{B} \times \mathbb{B}$ is rather different from $\mathbb{B}$; e.g. it is well-known that $\mathbb{B} \times \mathbb{B}$ adds Cohen reals whereas $\mathbb{B}$ does not (see [Je 2, part I, 5.9]). As it is known that some other forcing notions adding both Cohen and random reals (like $\mathbb{B} \times \mathbb{C} \cong \mathbb{B} * \mathbb{C}$ and $\mathbb{C} * \mathbb{B}$, where $\mathbb{C}$ is the Cohen algebra, and $*$ denotes iteration) do not add perfect sets of random reals (see [JS 2, 2.3] for $\mathbb{B} \times \mathbb{C}$ and [BJ 1, Theorem 2.13] for $\mathbb{C} * \mathbb{B}$), we may ask whether $\mathbb{B} \times \mathbb{B}$ does. We shall show in § 2 that the answer is again negative.

THEOREM 3: *$\mathbb{B} \times \mathbb{B}$ does not add a perfect set of random reals.*

The argument for this proof (which uses ideas from the proof that $\mathbb{B}$ does not add a perfect set of random reals — see [BJ 1, 2.1 - 2.4]) is rather long and technical; and one might get a shorter proof if the following question has a positive answer.

QUESTION 1: *Is $\mathbb{B} \times \mathbb{B}$ a complete subalgebra of $\mathbb{C} * \mathbb{B}$?*

We note here that all other embeddability relations between these three algebras adding both Cohen and random reals (namely, $\mathbb{B} \times \mathbb{C}$, $\mathbb{B} \times \mathbb{B}$, $\mathbb{C} * \mathbb{B}$) are known. We shall sketch the arguments which cannot be found in literature in 3.1.

Two further open problems are closely tied up with the Bartoszyński–Judah Theorem (*) and our Theorem 1, respectively.

QUESTION 2: *Given models of ZFC $M \subseteq N$ such that N contains both a dominating real and a random real over M, is there a perfect set of random reals over M in N?*

We shall show in 3.2 that to answer Question 2 it suffices to consider the problem whether $\mathbb{B} \times \mathbb{D} \cong \mathbb{B} * \mathbb{D}$ adds a perfect set of random reals, where $\mathbb{D}$ is Hechler forcing. We note that for many p.o.s $\mathbb{P}$ adding a dominating real (e.g. Mathias forcing) it is true that $\mathbb{B} \times \mathbb{P}$ adds a perfect set of random reals (3.3).

QUESTION 3: *Given models of ZFC $M \subseteq N$, does the existence of a perfect set of random reals over M in N imply the existence of an unbounded real over M in N?*

Here we say that $g \in \omega^\omega \cap N$ is an *unbounded real over M* iff $\forall f \in \omega^\omega \cap M\ \exists^\infty n\ (g(n) > f(n))$, where $\exists^\infty n$ means *there are infinitely many n* (dually, $\forall^\infty n$ abbreviates *for all but finitely many n*).

Motivation: One of the reasons for studying perfect sets of random reals concerns finite support iterations of *ccc* forcing notions. Namely, let $\langle \mathbb{P}_n, \breve{Q}_n;\ n \in \omega\rangle$ be an ω-stage finite support iteration such that for all $n \in \omega$, $\Vdash_{\mathbb{P}_n}$ "$\breve{Q}_n$ is *ccc*". Then the following are equivalent [JS 2, Theorem 2.1]:

(i) *There exists* $r \in V[G_\omega] \setminus \bigcup_n V[G_n]$ *random over* V,

(ii) *there exists* $n \in \omega$ *and* $T \in V[G_n]$ *a perfect set of random reals over* V,

where $\langle G_i;\ i \leq \omega\rangle$ is a chain of $\mathbb{P}_i$-generic filters. So *adding a random real in the ω-th stage* is stronger than just *adding a random real in an initial step* (on the other hand, $\mathbb{P}_\omega$ *adds a dominating real* is simply equivalent to *there is an* $n \in \omega$ *such that* $\mathbb{P}_n$ *adds a dominating real* [JS 1, Theorem 2.2]). Also perfect sets of random reals seem to play an important role in the investigation of the problem, posed by Fremlin, whether the smallest covering of the real line by measure zero sets can have cofinality ω. To build a model of ZFC where this is true we suggest an iterated forcing construction (with finite supports) which firstly adds ω_ω many Cohen reals over L to produce a family of ω_ω null sets which will still cover the real line in the final extension, and then goes through every subalgebra of the random algebra which is the random algebra restricted to some small inner model (in which the continuum has size $< \omega_\omega$) in $\omega_{\omega+1}$ steps (see the introduction of [JS 3] for details). By construction, we destroy all small covering families. So the main problem is to show that we do not add a real which does not belong to the family of ω_ω null sets added in the intermediate stage. To do this, it suffices (essentially) to prove that the whole iteration does not add a perfect set of random reals over the ground model L. We think that our Theorem 3 is a small but important step in this direction, and we hope that the ideas involved can be generalized to give a positive answer to

QUESTION 4: *Let* $\mathbf{A}$ *be a complete subalgebra of* $\mathbb{B}$. *Let* $\breve{\mathbb{B}}_{\mathbf{A}}$ *be an* $\mathbf{A}$*-name for* $\mathbb{B}$. *Is it true that* $\mathbb{B} * \breve{\mathbb{B}}_{\mathbf{A}}$ *does not add a perfect set of random reals?*

Cichoń's Theorem [BJ 1, Theorem 2.1] says that this is true if $\mathbf{A} = \mathbb{B}$, and our Theorem 3 gives a positive answer in case $\mathbf{A}$ is trivial.

Notation: Our notation is fairly standard. We refer the reader to [Je 1] for set theory and to [Ox] for measure theory. Most of the cited material will appear in the forthcoming book [BJ 2]. We now explain some notions which might be less familiar.

Given a finite sequence s (i.e. either $s \in 2^{<\omega}$ or $s \in \omega^{<\omega}$), we let $lh(s) :=$

dom(s) denote the length of s; for $\ell \in lh(s)$, $s\restriction\ell$ is the restriction of s to ℓ. $\hat{}$ is used for concatenation of sequences; and $\langle\rangle$ is the empty sequence. Furthermore, for $s \in 2^{<\omega}$, $[s] := \{f \in 2^\omega;\ f\restriction lh(s) = s\}$ is the set of branches through s (the open subset of 2^ω determined by s).

Given a perfect tree $T \subseteq 2^{<\omega}$ and $s \in T$, we let $T_s := \{t \in T;\ t \subseteq s$ or $s \subseteq t\}$; and stem(T) $:= \bigcup\{s \in T;\ T_s = T\}$ is the stem of T. For $\ell \in \omega$, we let $T\restriction\ell := \{s \in T;\ lh(s) \leq \ell\}$, the finite initial part of T of height ℓ. We will confuse finite trees T with all branches of fixed length ℓ with the set of branches $[T] := \{s \in T;\ lh(s) = \ell\}$.

We assume the reader to be familiar with forcing and Boolean-valued models (see [Je 1], [Je 2]). We suppose that all our p.o.s (forcing notions) have a largest element 1 . Given a p.o. $\mathbb{P} \in V$, we shall denote $\mathbb{P}$-names by symbols like $\check{f}$, $\check{T}$, ... and their interpretation in $V[G]$ (where G is $\mathbb{P}$-generic over V) by $\check{f}[G]$, $\check{T}[G]$, ... If φ is a sentence of the $\mathbb{P}$-forcing language, we let $\parallel \varphi \parallel$ be the Boolean value of φ; i.e. the maximal element forcing φ in the complete Boolean algebra $r.o.(\mathbb{P})$ associated with $\mathbb{P}$. We shall often confuse $\mathbb{P}$ and $r.o.(\mathbb{P})$.

We equip $\mathbb{B} \times \mathbb{B} := \{(p,q);\ p,q \in \mathbb{B}\setminus\{0\}\} \cup \{0\}$ with the product measure (i.e. $\mu(p,q) = \mu(p)\cdot\mu(q)$). Then $\mu: \mathbb{B}\times\mathbb{B} \to [0,1]$ is finitely additive and strictly positive (any non-zero condition has positive measure). By [Ka, Proposition 2.1], μ can be extended to a finitely additive, strictly positive measure on $r.o.(\mathbb{B}\times\mathbb{B})$. This will be used in 2.5. Note that this measure is not σ-additive.

1. Not adding dominating reals

1.1 We shall now introduce the framework needed to prove Theorem 1. Besides giving the latter result this framework will also provide us with preservation results for *not adding dominating reals* in finite support products and finite support iterations.

Let $\mathbb{P}$ be an arbitrary p.o. A function $h: \mathbb{P} \to \omega$ is a **height function** iff $p \leq q$ implies $h(p) \geq h(q)$. A pair $(\mathbb{P}, h)$ is **soft** iff $\mathbb{P}$ is a p.o., h is a height function on $\mathbb{P}$, and the following two conditions are satisfied:

(I) (decreasing chain property) if $\{p_n;\ n \in \omega\}$ is decreasing and $\exists m \in \omega\ \forall n \in \omega\ (h(p_n) \leq m)$, then $\exists p \in \mathbb{P}\ \forall n \in \omega\ (p \leq p_n)$;

(II) (weak finite cover property) given $m \in \omega$ and $\{p_i;\ i \in n\} \subseteq \mathbb{P}$ there is $\{q_j;\ j \in k\} \subseteq \mathbb{P}$ so that

(i) $\forall i \in n,\ j \in k$, q_j is incompatible with p_i;

(ii) whenever q is incompatible with all p_i and $h(q) \leq m$ then there exists $j \in k$ so that $q \leq q_j$.

We also consider the following property of pairs $(\mathbb{P}, h)$ — where $\mathbb{P}$ is a p.o. and h a height function on $\mathbb{P}$:

(*) given a maximal antichain $\{p_n;\ n \in \omega\} \subseteq \mathbb{P}$ and $m \in \omega$ there exists $n \in \omega$ such that: whenever p is incompatible with $\{p_j;\ j \in n\}$ then $h(p) > m$.

1.2 Lemma: *If $(\mathbb{P}, h)$ is soft, then $(\mathbb{P}, h)$ has property (*).*

Proof: Suppose not and let $\{p_n;\ n \in \omega\}$ and $m \in \omega$ witness the contrary. For each $n \in \omega$ let $\{q_j^n;\ j \in k_n\}$ be a weak finite cover with respect to $\{p_i;\ i \in n\}$, m according to (II). By assumption none of these sets can be empty and we can assume that each q_j^n has height $\leq m$. By the cover property (II) (ii) they form an ω-tree with finite levels with respect to "$\leq$". By König's Lemma this tree has an infinite branch. By (I) there is a condition below this branch, contradicting the fact that $\{p_n;\ n \in \omega\}$ is a maximal antichain. ∎

1.3 Theorem: *Suppose $\mathbb{P}$ is a ccc p.o., h is a height function on $\mathbb{P}$, and $(\mathbb{P}, h)$ satisfies property (*). Then any unbounded family of functions in $\omega^\omega \cap V$ is still unbounded in $V[G]$, where G is $\mathbb{P}$-generic over V.*

Proof: Let F be unbounded in $\omega^\omega \cap V$. Suppose $\Vdash_{\mathbb{P}} \breve{f} \in \omega^\omega$. For each $m \in \omega$ let $\{p_n^m;\ n \in \omega\}$ be a maximal antichain deciding the value $\breve{f}(m)$. Choose n_m according to (*) so that: whenever p is incompatible with $\{p_j^m; j \in n_m\}$, then $h(p) > m$. Define $f\colon \omega \to \omega$ by setting $f(m) :=$ the maximum of the values of $\breve{f}(m)$ decided by $\{p_j^m; j \in n_m\}$. Let $g \in F$ be a function which is not dominated by f. We claim that $\Vdash_{\mathbb{P}}$ "$\breve{f}$ does not dominate g".

For suppose there is a $p \in \mathbb{P}$ and a $k \in \omega$ so that

$$p \Vdash \forall m \geq k \ \ \breve{f}(m) > g(m).$$

Choose $m \geq k$ so that $h(p) \leq m$ and $g(m) \geq f(m)$. Then p must be compatible with p_j^m for some $j \in n_m$. But if q is a common extension, then

$$q \Vdash \breve{f}(m) > g(m) \geq f(m) \geq \breve{f}(m),$$

a contradiction. ∎

1.4 Towards the proof of Theorem 1: We think of $\mathbb{B}$ as consisting of sets $B \subseteq 2^\omega$ of positive measure so that for all $t \in 2^{<\omega}$, if $[t] \cap B \neq \emptyset$ then $\mu([t] \cap B) > 0$; for $m \in \omega$ let $B \cap 2^m := \{t \in 2^m;\ [t] \cap B \neq \emptyset\}$. Then we define the following p.o. $(\mathbb{P}, \leq)$:

$$(B,n) \in \mathbb{P} \iff B \in \mathbb{B} \ \wedge \ n \in \omega;$$

$$(B,n) \leq (C,m) \iff B \subseteq C \ \wedge \ n \geq m \ \wedge \ B \cap 2^m = C \cap 2^m.$$

It follows from the *ccc*-ness of any product of finitely many copies [Je 2, part I, 5.7] of $\mathbb{B}$ that $\mathbb{P}$ is *ccc*, too. Clearly, $\mathbb{P}$ generically adds a perfect set of random reals, and we have to show that it does not add dominating reals. To this end, we will introduce a height function on $\mathbb{P}$.

In fact, let $\mathbb{P}' \subseteq \mathbb{P}$ be the set of conditions $(B,n) \in \mathbb{P}$ so that for all s in $2^n \cap B$, $\mu([s] \cap B) \geq 2^{-(lh(s)+1)}$. $\mathbb{P}'$ is dense in $\mathbb{P}$ (by the Lebesgue density Theorem [Ox, Theorem 3.20]). We define $h\colon \mathbb{P}' \to \omega$ by $h((B,n)) = n$ and work with $\mathbb{P}'$ from now on.

1.5 LEMMA: *$(\mathbb{P}', h)$ is soft.*

Remark: By 1.2 and 1.3 the proof of this Lemma finishes the proof of Theorem 1.

Proof: (I) is clear (for if $\{(B_n, m);\ n \in \omega\}$ is decreasing then $(\cap B_n, m)$ is a lower bound because we took our conditions from $\mathbb{P}'$). For (II) we use:

1.6 MAIN CLAIM: *Given (B,n), $(C,m) \in \mathbb{P}'$ and $k \in \omega$ there are finitely many conditions $\{q_i;\ i \in j\}$ below (C,m) so that*

(i) *each q_i is incompatible with (B,n);*

(ii) *if q is incompatible with (B,n), $h(q) \leq k$, and $q \leq (C,m)$, then $\exists i \in j$ $(q \leq q_i)$.*

Proof: Without loss $k \geq m,n$. Assume $n \geq m$. Let ℓ be such that $m \leq \ell \leq n$. We now describe which conditions of height ℓ we put into our finite set.

(i) For each $T \subseteq 2^\ell$ with $T\restriction m = C \cap 2^m$ and $T \subseteq C \cap 2^\ell$ and $T \neq B \cap 2^\ell$ let $C_T \in \mathbb{B}$ be such that $C_T \cap 2^\ell = T$ and $C_T \cap [t] = C \cap [t]$ for each $t \in T$. If $(C_T, \ell) \in \mathbb{P}'$, then put (C_T, ℓ) into the set.

(ii) For each $T \subseteq 2^n$ with $T\restriction m = C \cap 2^m$ and $T \subseteq C \cap 2^n$ and $T\restriction \ell = B \cap 2^\ell$ and $T \not\supseteq B \cap 2^n$ let $C_T \in \mathbb{B}$ be such that $C_T \cap 2^n = T$ and $C_T \cap [t] = C \cap [t]$ for each $t \in T$. If $(C_T, \ell) \in \mathbb{P}'$, then put (C_T, ℓ) into the set.

(iii) For each $T \subseteq 2^n$ with $T \upharpoonright m = C \cap 2^m$ and $T \subseteq C \cap 2^n$ and $T \upharpoonright \ell = B \cap 2^\ell$ and $T \supseteq B \cap 2^n$ and for each $t \in B \cap 2^n$ let $C_{T,t} \in \mathbb{B}$ be such that $C_{T,t} \cap 2^n = T$ and $C_{T,t} \cap [s] = C \cap [s]$ for each $s \in T \setminus \{t\}$ and $C_{T,t} \cap [t] = (C \cap [t]) \setminus B$. If $(C_{T,t}, \ell) \in \mathbb{P}'$, then put $(C_{T,t}, \ell)$ into the set.

It is easy to see that any condition of height ℓ below (C, m) which is incompatible with (B, n) lies below one the conditions defined in (i) – (iii) above.

Next suppose that $n \leq \ell \leq k$. Then we can again find a finite set of conditions of height ℓ satisfying the requirements of the main claim for conditions of height ℓ by an argument similar to the one in (i) – (iii) above.

This takes care of the case when $n \geq m$. So assume now $n \leq m$. Then we get our set of conditions as in the preceding paragraph. ∎

1.7 Proof of (II) of Lemma 1.5 from the main claim 1.6: We make induction using the main claim repeatedly. I.e. let $(B, n) = p_0$ and $(C, m) = (2^\omega, 0)$ and apply the main claim to them to get $\{q_i;\ i \in j\}$. Then let $(B, n) = p_1$ and $(C, m) = q_i$ $(i \in j)$ and apply the main claim j times to get a new family. Etc. ∎

This finishes the proof of Lemma 1.5 and of Theorem 1. ∎

1.8 THEOREM:

(i) *Suppose* $\langle \mathbb{P}_\alpha, (\check{Q}_\alpha, \check{h}_\alpha);\ \alpha < \kappa \rangle$ *is a finite support iteration of arbitrary length* κ *(*κ *limit) such that*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\check{Q}_\alpha \textit{ is ccc}, \check{h}_\alpha \textit{ is a height function on } \check{Q}_\alpha \textit{ and } (\check{Q}_\alpha, \check{h}_\alpha) \textit{ satisfies } (*)\text{”}.$$

Then $\mathbb{P}_\kappa = \lim_{\alpha<\kappa} \mathbb{P}_\alpha$ *does not add dominating reals.*

(ii) *Suppose* $\langle (\mathbb{P}_\alpha, h_\alpha);\ \alpha < \kappa \rangle$ *is a sequence of soft ccc p.o.s of arbitrary length* κ*. Then there is a height function* h *on the finite support product* $\mathbb{P}$ *of the* $\mathbb{P}_\alpha$ $(\alpha < \kappa)$ *so that* $(\mathbb{P}, h)$ *is soft.*

Remark: In particular, both the finite support iteration and the finite support product of an arbitrary number of ccc soft p.o.s does not add dominating reals (cf 1.2, 1.3).

Proof: (i) It suffices to show by induction on α that

$$\Vdash_{\mathbb{P}_\alpha} \omega^\omega \cap V \text{ is unbounded in } \omega^\omega.$$

If α is a limit ordinal, this follows from [JS 1, Theorem 2.2]. So suppose α is a successor. Then the result follows from Theorem 1.3 and the induction hypothesis.

(ii) We make again induction on α. Let $\mathbb{Q}_\alpha$ be the finite support product of the $\mathbb{P}_\beta$ where $\beta < \alpha$. We shall recursively construct height functions $g_\alpha\colon \mathbb{Q}_\alpha \to \omega$ such that

(a) for $\alpha < \beta$, $g_\alpha \subseteq g_\beta$;
(b) $g_\alpha(q) \geq \max_{\beta<\alpha} h_\beta(q\restriction\mathbb{P}_\beta)$;
(c) $g_\alpha(q) \geq |\text{supp}(q)|$;
(d) $(\mathbb{Q}_\alpha, g_\alpha)$ is soft.

Clearly, a g_α satisfying (b) and (c) will satisfy the decreasing chain property (I) as well. (We assume without loss that $\forall\beta < \kappa$, $h_\beta(1\ ) = 0$.)

We first deal with the case when α is a successor ordinal, $\alpha = \beta + 1$. Then $\mathbb{Q}_\alpha = \mathbb{Q}_\beta \times \mathbb{P}_\beta$. Let $m := \max\{g_\beta(q), h_\beta(p)\}$ and define $g_\alpha\colon \mathbb{Q}_\alpha \to \omega$ by

$$g_\alpha(q,p) := \begin{cases} m+1 & \text{if } |\text{supp}(q)| = m \text{ and } p \neq 1 \\ m & \text{otherwise} \end{cases}$$

for $(q,p) \in \mathbb{Q}_\beta \times \mathbb{P}_\beta$. g_α is a height function on $\mathbb{Q}_\alpha$ which is easily seen to satisfy (a) — (c) above.

To show that $(\mathbb{Q}_\alpha, g_\alpha)$ satisfies the weak finite cover property (II), let $\{(q_i,p_i);\ i \in n\}$ be a finite subset of $\mathbb{Q}_\alpha$ and let $m \in \omega$. For each $A \subseteq n$ let $\{q_j^A;\ j \in k_A\}$ be a weak finite cover with respect to $\{q_i;\ i \in A\}$ and m in $\mathbb{Q}_\beta$ (i.e. (i) $\forall i \in A$, $j \in k_A$, q_j^A is incompatible with q_i; and (ii) whenever q is incompatible with all q_i ($i \in A$) and $h(q) \leq m$ then there exists $j \in k_A$ so that $q \leq q_j^A$), and let $\{p_j^A;\ j \in \ell_A\}$ be a weak finite cover with respect to $\{p_i;\ i \in n \setminus A\}$ and m. We claim that the family $F := \{(q_i^A, p_j^A);\ A \subseteq n\ \wedge\ i \in k_A\ \wedge\ j \in \ell_A\}$ is a weak finite cover with respect to $\{(q_i,p_i);\ i \in n\}$ and m.

Clearly, F satisfies (i) of the definition of the weak finite cover property (II). Furthermore, if (q,p) is incompatible with all (q_i,p_i) ($i \in n$) there exists $A \subseteq n$ such that q is incompatible with all q_i for $i \in A$ and p is incompatible with all p_i for $i \in n \setminus A$. So if $g_\alpha(q,p) \leq m$ (in particular, $g_\beta(q) \leq m$ and $h_\beta(p) \leq m$) then we can find $j \in k_A$ and $j' \in \ell_A$ such that $q \leq q_j^A$ and $p \leq p_{j'}^A$; i.e. $(q,p) \leq (q_j^A, p_{j'}^A)$. This shows (ii) in the definition of the weak finite cover property (II).

Now suppose α is a limit ordinal. Then let $g_\alpha := \bigcup_{\beta<\alpha} g_\beta$. g_α clearly satisfies (a) — (c), and the weak finite cover property (II) for $(\mathbb{Q}_\alpha, g_\alpha)$ follows from the weak finite cover properties of the $(\mathbb{Q}_\beta, g_\beta)$ for $\beta < \alpha$ (because (II) talks only about finitely many conditions). ∎

1.9 We note here that the notions discussed so far are closely tied up with some cardinal invariants of the continuum. Namely, we let $\mathcal{N}$ denote the ideal of null sets and

add($\mathcal{N}$) := the least κ such that $\exists \mathcal{F} \in [\mathcal{N}]^\kappa$ $(\bigcup \mathcal{F} \notin \mathcal{N})$;

$w$cov($\mathcal{N}$) := the least κ such that $\exists \mathcal{F} \in [\mathcal{N}]^\kappa$ ($2^\omega \setminus \bigcup \mathcal{F}$ does not contain a perfect set);

cov($\mathcal{N}$) := the least κ such that $\exists \mathcal{F} \in [\mathcal{N}]^\kappa$ $(\bigcup \mathcal{F} = 2^\omega)$;

unif($\mathcal{N}$) := the least κ such that $[2^\omega]^\kappa \setminus \mathcal{N} \neq \emptyset$;

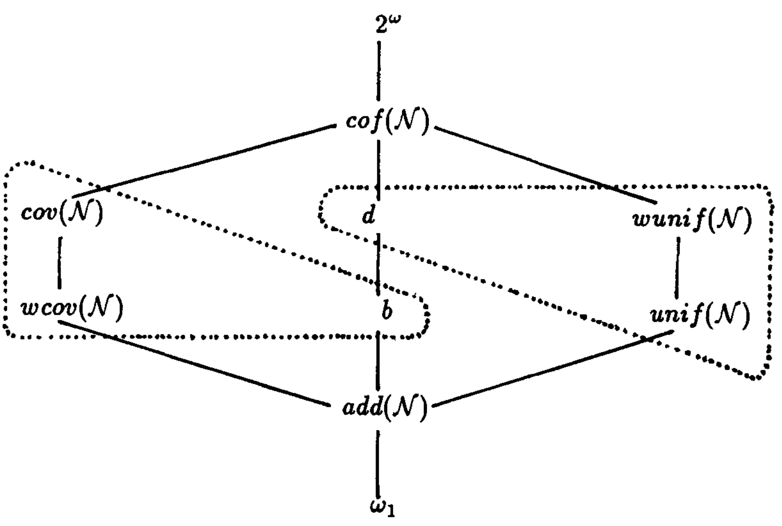
$w$unif($\mathcal{N}$) := the least κ such that there is a family $\mathcal{F} \in [[2^{<\omega}]^\omega]^\kappa$ of perfect sets with $\forall N \in \mathcal{N}\, \exists T \in \mathcal{F}\, (N \cap T = \emptyset)$;

cof($\mathcal{N}$) := the least κ such that $\exists \mathcal{F} \in [\mathcal{N}]^\kappa\, \forall A \in \mathcal{N}\, \exists B \in \mathcal{F}\, (A \subseteq B)$;

b := the least κ such that $\exists \mathcal{F} \in [\omega^\omega]^\kappa\, \forall f \in \omega^\omega\, \exists g \in \mathcal{F}\, \exists^\infty n\, (g(n) > f(n))$;

d := the least κ such that $\exists \mathcal{F} \in [\omega^\omega]^\kappa\, \forall f \in \omega^\omega\, \exists g \in \mathcal{F}\, \forall^\infty n\, (g(n) > f(n))$.

Then we can arrange these cardinals in the following diagram.



Here the invariants get larger as one moves up in the diagram. $b \geq \text{add}(\mathcal{N})$ (and dually $d \leq \text{cof}(\mathcal{N})$) is due to Miller [Mi]. The dotted line says that $w\text{cov}(\mathcal{N}) \geq \min\{\text{cov}(\mathcal{N}), b\}$ (and dually, $w\text{unif}(\mathcal{N}) \leq \max\{\text{unif}(\mathcal{N}), d\}$). This can be seen from the Bartoszyński–Judah result (*) in the Introduction as follows. Suppose $\lambda := w\text{cov}(\mathcal{N}) < \min\{\text{cov}(\mathcal{N}), b\}$. Let M be a model of enough ZFC of size λ containing a weak covering family. As $\lambda < b$ there is a real $f \in \omega^\omega$ dominating all reals in M. Let N be a model of enough ZFC of size λ containing M and f. As $\lambda < \text{cov}(\mathcal{N})$, there is a real $r \in 2^\omega$ random over N. By (*) this implies that there is a perfect set of random reals over M, a contradiction. — Iterating the p.o. from Theorem 1 we get:

THEOREM 1': *For any regular cardinal κ, it is consistent that $w\text{cov}(\mathcal{N}) = \kappa$ while $b = \omega_1$; dually, it is consistent that $w\text{unif}(\mathcal{N}) = \omega_1$ while $d = \kappa$.*

Proof: (a) Assume CH. We make a finite support iteration of length κ of the p.o. $\mathbb{P}$ described in 1.4. In the generic extension we have $w\text{cov}(\mathcal{N}) = \kappa$ because we added κ many perfect sets of random reals; and $b = \omega_1$ by 1.5, 1.2 and 1.8 (i).

(b) Assume $MA + 2^\omega = \kappa$; and make a finite support iteration of length ω_1 of $\mathbb{P}$. Again standard arguments show that $w\text{unif}(\mathcal{N}) = \omega_1$ and $d = \kappa$ in the generic extension. ∎

The most interesting open question concerning the relationship between these cardinals is connected with Question 3 in the Introduction.

QUESTION 3': *Is it consistent that $w\text{cov}(\mathcal{N}) > d$? Dually, is it consistent that $w\text{unif}(\mathcal{N}) < b$?*

1.10 Proof of Theorem 2: By 1.2, 1.3 and 1.8 (ii) it suffices to show that there is a height function $h\colon \mathbb{B} \to \omega$ so that $(\mathbb{B}, h)$ is soft. But this is easy: for $B \in \mathbb{B}$ let $h(B) := \min\{n \in \omega;\ \mu(B) \geq \frac{1}{n}\}$. ∎

We note that this height function h and also the height function it induces on $\mathbb{B} \times \mathbb{B}$ by 1.8 (ii) have a *strong* finite cover property: (ii) in (II) can be replaced by: whenever q is incompatible with all p_i then there exists $j \in k$ so that $q \leq q_j$.

2. Not adding perfect sets of random reals

2.1 This whole section is devoted to the proof of Theorem 3. Lemmata 2.2 and 2.3 below which we single out from the principal argument bear the imprint of the proof of Cichoń's Theorem in [BJ 1, 2.1 – 2.4]. The main new idea comes in in 2.4. The rest (2.5 – 2.9) is mostly technical.

Given $k', k \in \omega$, $k' < k$, we let

$$\epsilon_{k,k'} := 2^{1-k} \cdot (1 + \binom{k}{1} + \cdots + \binom{k}{k'-1}).$$

Clearly, given any $k' \in \omega$, we can find $k > k'$ so that $\epsilon_{k,k'}$ is arbitrarily small.

2.2 LEMMA: *Given $n, k, k' \in \omega$ ($k' \leq k$ and $k \leq 2^n$) and $Z \subseteq 2^n$ and real numbers a_T for each $T \subseteq 2^n$ with $|T \cap Z| \geq k$ there exists $Z' \subseteq Z$ of size $\leq |Z|/2$ such that $\sum_{|T\cap Z'|\geq k'} a_T \geq \sum_T a_T \cdot (1 - \epsilon_{k,k'})$.*

Proof: Let $a := \sum_T a_T$. For any λ close to 1 ($\lambda < 1$) we can choose $\ell \in \omega$ and $\{T_i;\ i \in \ell\}$ so that

$$a_T \cdot \lambda < \frac{|\{i;\ T_i = T\}|}{\ell} \cdot a < a_T \cdot \lambda^{-1}$$

for all T. For $i \in \ell$ let $\mathcal{Z}_i := \{Z' \subseteq Z;\ |Z' \cap T_i| \geq k' \text{ and } |(Z \setminus Z') \cap T_i| \geq k'\}$. Then $|\mathcal{Z}_i| \cdot 2^{-|Z|} \geq 1 - \epsilon_{k,k'}$ for all $i \in \ell$. We claim that there is an $X \subseteq \ell$ of size $\geq \ell \cdot (1 - \epsilon_{k,k'})$ so that $\bigcap_{i\in X} \mathcal{Z}_i \neq \emptyset$.

For suppose not. Then for each $Z' \subseteq Z$, $|\{i \in \ell;\ Z' \in \mathcal{Z}_i\}| < \ell \cdot (1 - \epsilon_{k,k'})$. Hence

$$2^{|Z|} \cdot \ell \cdot (1 - \epsilon_{k,k'}) \leq \sum_{i\in\ell} |\mathcal{Z}_i| = \sum_{Z'\subseteq Z} |\{i \in \ell;\ Z' \in \mathcal{Z}_i\}| < 2^{|Z|} \cdot \ell \cdot (1 - \epsilon_{k,k'}),$$

a contradiction.

Now choose $Z' \in \bigcap_{i\in X} \mathcal{Z}_i$. Then either $|Z'| \leq |Z|/2$ or $|Z \setminus Z'| \leq |Z|/2$. Assume without loss that $|Z'| \leq |Z|/2$. Furthermore

$$\sum_{|T\cap Z'|\geq k'} a_T \cdot \lambda^{-1} > \sum_{|T\cap Z'|\geq k'} \frac{|\{i;\ T_i = T\}|}{\ell} \cdot a \geq \frac{|X|}{\ell} \cdot a \geq a \cdot (1 - \epsilon_{k,k'}).$$

Because there are only finitely many possibilities for the sum on the lefthand side, we can choose λ so small that for the Z' chosen according to this λ we have

$$\sum_{|T\cap Z'|\geq k'} a_T \geq a \cdot (1 - \epsilon_{k,k'}).$$

This finishes the proof of the Lemma. ∎

2.3 Lemma: *Given a real $\epsilon > 0$ and $m \in \omega$ the following is true for large enough $k, n \in \omega$: given real numbers a_T for each $T \subseteq 2^n$ with $|T| \geq k$ there exists a $Z \subseteq 2^n$ of size $\leq 2^{n-m}$ such that $\sum_{T \cap Z \neq \emptyset} a_T \geq \sum_T a_T \cdot (1-\epsilon)$.*

Remark: We say that a statement is true for *large enough* n iff $\exists k \in \omega$ so that $\forall n \geq k$ the statement is true.

Proof: Construct recursively a sequence $\langle k_i;\ i \leq m\rangle$ of natural numbers so that $\prod_{i \in m}(1 - \epsilon_{k_{i+1}, k_i}) \geq (1-\epsilon)$ where $k_0 = 1$. Let $k \geq k_m$ and n so large that $k \leq 2^n$. Now apply Lemma 2.2 m times to get Z. ∎

2.4 Diagonal chains: It turns out that a detailed investigation of antichains in $\mathbb{B} \times \mathbb{B}$ is necessary for the proof of Theorem 3. We say $(p, q) \in \mathbb{B} \times \mathbb{B}$ is **quadratic** iff $\mu(p) = \mu(q)$. Clearly the quadratic conditions are dense in $\mathbb{B} \times \mathbb{B}$ so that it suffices (essentially) to consider them. More generally, given $(p, q) \in \mathbb{B} \times \mathbb{B}$, (p', q') is **quadratic in** (p, q) iff $p' \leq p$, $q' \leq q$ and $\mu(p')/\mu(p) = \mu(q')/\mu(q)$.

$\{(p_n, q_n);\ n \in \omega\}$ is said to be a **first order diagonal chain** in $\mathbb{B} \times \mathbb{B}$ iff

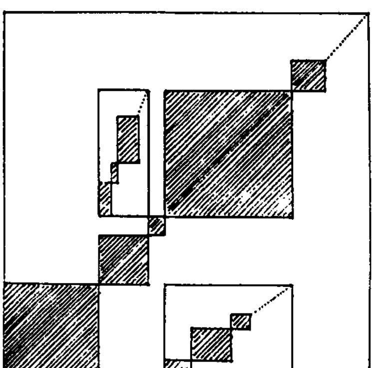
(1) each (p_n, q_n) is quadratic;

(2) both $\{p_n;\ n \in \omega\}$ and $\{q_n;\ n \in \omega\}$ are maximal antichains in $\mathbb{B}$.

More generally we say that $C = \{(p_n^{\sigma\tau}, q_n^{\sigma\tau});\ n \in \omega,\ \sigma, \tau \in \omega^{<m},\ lh(\sigma) = lh(\tau),\ \forall i \in lh(\sigma)\ (\sigma(i) \neq \tau(i))\}$ is an m**th order diagonal chain** in $\mathbb{B} \times \mathbb{B}$ (for $m \geq 2$) iff $\{(p_n^{\sigma\tau}, q_n^{\sigma\tau}) \in C;\ lh(\sigma) < m-1\}$ is an $(m-1)$th order diagonal chain and for each σ, τ of length $m-1$ with $\tau(m-2) \neq \sigma(m-2)$,

(1) $\forall n \in \omega$, $(p_n^{\sigma\tau}, q_n^{\sigma\tau})$ is quadratic in $(p_{\sigma(m-2)}^{\sigma\restriction(m-2)\tau\restriction(m-2)}, q_{\tau(m-2)}^{\sigma\restriction(m-2)\tau\restriction(m-2)})$;

(2) $\{p_n^{\sigma\tau};\ n \in \omega\}$ is a maximal antichain of conditions below $p_{\sigma(m-2)}^{\sigma\restriction(m-2)\tau\restriction(m-2)}$ in $\mathbb{B}$ and $\{q_n^{\sigma\tau};\ n \in \omega\}$ is a maximal antichain of conditions below the condition $q_{\tau(m-2)}^{\sigma\restriction(m-2)\tau\restriction(m-2)}$ in $\mathbb{B}$.



A second order diagonal chain

Clearly, below any antichain A in $\mathbb{B} \times \mathbb{B}$ there is an mth order diagonal chain C for each m (in the sense that any condition in C is below some condition in A).

2.5 Towards the proof of Theorem 3: Let $\check{T}$ be a $\mathbb{B} \times \mathbb{B}$-name so that

$$\Vdash_{\mathbb{B}\times\mathbb{B}} \text{“}\check{T} \text{ is perfect”}.$$

We want to construct a null set N in the ground model so that

$$\Vdash_{\mathbb{B}\times\mathbb{B}} \text{“}N \cap \check{T} \neq \emptyset\text{”}.$$

More explicitly, using Lemma 2.3, we shall construct sequences $\langle n_m;\ m \in \omega\rangle$ and $\langle Z_m;\ m \in \omega\rangle$ so that $Z_m \subseteq 2^{n_m}$, $|Z_m| \leq 2^{n_m - m}$ and

$$\Vdash_{\mathbb{B}\times\mathbb{B}} \exists x \in \check{T}\ \exists^\infty m\ (x \restriction n_m \in Z_m).$$

This will imply the required result for $N := \{x \in 2^\omega;\ \exists^\infty m\ (x\restriction n_m \in Z_m)\}$ is a null set (see below in 2.9). We set $n_0 := 0$ and $Z_0 := \emptyset$. Now let $m > 0$ and assume that n_{m-1} and Z_{m-1} have been defined. We shall describe the construction of n_m and Z_m.

Let $\delta > 0$ be very small; let $\langle z_j;\ j \in m\rangle$, $\langle y_j;\ j \in m-1\rangle$ be sequences of natural numbers so that $z_0 > 1$, $y_j = 4 \cdot z_j$ and y_j / z_{j+1} is very small; let $\epsilon > 0$ such that $\delta^{-2} \cdot \epsilon \cdot m \cdot z_{m-1}$ is very small (in fact, we want that

$$\zeta_m = \zeta = 2 \cdot m \cdot (\epsilon + \delta^{-2} \cdot \epsilon \cdot m \cdot z_{m-1} + \delta) + \sum_{j=0}^{m-2} \frac{y_j}{z_{j+1}}$$

is – say – smaller than $1/4^m$; cf 2.6); let $v \in \omega$ be such that $z_{m-1} \leq v \cdot (1-\epsilon)^2$. Choose k according to Lemma 2.3 for ϵ and $m + 2 \cdot n_{m-1}$. Let $\{(p_i^{\sigma\tau}, q_i^{\sigma\tau});\ i \in \omega,\ \sigma, \tau \in \omega^{<m},\ lh(\sigma) = lh(\tau),\ \forall j \in lh(\sigma)\ (\sigma(j) \neq \tau(j))\}$ be an mth order diagonal chain in $\mathbb{B} \times \mathbb{B}$ deciding $\check{T}$ up to some level $n_i^{\sigma\tau}$ such that for each $s \in 2^{\leq n_{m-1}}$,

$$\text{either}\quad (p_i^{\sigma\tau}, q_i^{\sigma\tau}) \Vdash s \in \check{T} \wedge |\check{T}_s \restriction n_i^{\sigma\tau}| \geq k$$
$$\text{or}\quad (p_i^{\sigma\tau}, q_i^{\sigma\tau}) \Vdash s \notin \check{T}.$$

We can construct this diagonal chain in such a way that for σ, τ of length $\ell < m$ (with $\forall j \in \ell\ (\sigma(j) \neq \tau(j))$),

$$(5.1)\qquad \sum_i \mu(p_i^{\sigma\tau}, q_i^{\sigma\tau}) < \frac{1}{v} \cdot \mu(p_{\sigma(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)}, q_{\tau(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)})$$

(where we make the convention that for $\ell = 0$, $\sigma = \tau = \langle\rangle$,

$$p_{\sigma(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)} = q_{\tau(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)} = 1 \ ,$$

the maximal element of $\mathbb{B}$).

We now define recursively which pairs of sequences $\sigma\tau$ ($\sigma, \tau \in \omega^{<m}$, $lh(\sigma) = lh(\tau)$, $\forall j \in lh(\sigma)\ (\sigma(j) \neq \tau(j))$) are **relevant.** For relevant pairs we also define $a^{\sigma\tau} \in \mathbb{R}$ and $j^{\sigma\tau} \in \omega$. $\langle\rangle\langle\rangle$ is relevant. Choose $j^{\langle\rangle\langle\rangle} \in \omega$ such that $a^{\langle\rangle\langle\rangle} := \sum_{i \in j^{\langle\rangle\langle\rangle}} \mu(p_i^{\langle\rangle\langle\rangle}) \geq 1 - \epsilon$. Suppose $a^{\sigma\tau}$ and $j^{\sigma\tau}$ are defined for relevant pairs $\sigma\tau$ of length ℓ ($0 \leq \ell < m-1$). Then $\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle$ is relevant iff $i, j \in j^{\sigma\tau}$ and $i \neq j$. Furthermore, for each such i, j, choose $j^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} \in \omega$ such that

$$(5.2)\quad a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} := \sum_{i' \in j^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle}} \mu(q_{i'}^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle}) \cdot \mu(p_i^{\sigma\tau}) \geq \mu(p_i^{\sigma\tau}) \cdot \mu(q_j^{\sigma\tau}) \cdot (1-\epsilon).$$

Now let

$$n_m := n := \max_{\sigma\tau \text{ relevant}, i \in j^{\sigma\tau}} n_i^{\sigma\tau}.$$

Fix $s \in 2^{\leq n_{m-1}}$. For $T \subseteq 2^n$ with $|T| \geq k$ and $s \subseteq \text{stem}(T)$ and for relevant tuples $\sigma\tau$ let

$$(5.3)\quad a_T^{\sigma\tau} := \sum_{i \in j^{\sigma\tau}} \frac{\mu(\| s \in \check{T} \wedge \check{T}_s\restriction n = T \| \cap (p_i^{\sigma\tau}, q_i^{\sigma\tau}))}{\mu(p_i^{\sigma\tau})} \cdot \mu(p_{\sigma(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)}).$$

And let

$$a_s^{\sigma\tau} := \sum_{i \in j^{\sigma\tau}} \frac{\mu(\| s \notin \check{T} \| \cap (p_i^{\sigma\tau}, q_i^{\sigma\tau}))}{\mu(p_i^{\sigma\tau})} \cdot \mu(p_{\sigma(\ell-1)}^{\sigma\restriction(\ell-1)\tau\restriction(\ell-1)}).$$

Then $\sum_T a_T^{\sigma\tau} + a_s^{\sigma\tau} = a^{\sigma\tau}$ (this uses the finite additivity of the measure on $r.o.(\mathbb{B} \times \mathbb{B})$ – see Introduction). Let

$$a_T^j := \sum_{lh(\sigma)=lh(\tau)=j} a_T^{\sigma\tau}, \quad a_s^j := \sum_{lh(\sigma)=lh(\tau)=j} a_s^{\sigma\tau}$$
$$\text{and}\quad a^j := \sum_T a_T^j + a_s^j = \sum_{lh(\sigma)=lh(\tau)=j} a^{\sigma\tau}.$$

Let

$$a_T := \sum_{j \in m} \frac{a_T^j}{a^j - a_s^j}.$$

Apply Lemma 2.3 to get $Z^s_m := Z^s \subseteq 2^n$ of size $\leq 2^{n-m-2\cdot n_{m-1}}$ such that

$$\sum_{T\cap Z^s\neq\emptyset} a_T \geq \sum_T a_T\cdot(1-\epsilon) = m\cdot(1-\epsilon). \tag{5.4}$$

Finally, set $Z_m := Z := \bigcup_{s\in 2^{\leq n_{m-1}}} Z^s$. This completes the construction of $n_m = n$ and $Z_m = Z$.

2.6 Main Claim: *Let $s\in 2^{\leq n_{m-1}}$. Suppose (p,q) is a quadratic condition such that $(p,q)\Vdash$ "$s\in\check{T}\wedge Z^s_m\cap\check{T}_s=\emptyset$". Then $\mu(p,q) < \frac{1}{4^m}+\zeta_m$ (where*

$$\zeta_m = \zeta = 2\cdot m\cdot(\epsilon+\delta^{-2}\cdot\epsilon\cdot m\cdot z_{m-1}+\delta) + \sum_{j=0}^{m-2}\frac{y_j}{z_{j+1}}$$

as in 2.5).

Proof: The proof of the main claim will take some time (up to 2.8); to make our argument (which is essentially one big estimation) go through smoothly we need to make some conventions and introduce a few more notions.

If σ is a sequence of length $\ell\geq 1$, $\hat\sigma = \sigma\restriction(\ell-1)$ will be the sequence with the last value deleted. For $\ell = 0$,

$$p^{\hat{\langle\rangle}\hat{\langle\rangle}}_{\langle\rangle(\ell-1)} = q^{\hat{\langle\rangle}\hat{\langle\rangle}}_{\langle\rangle(\ell-1)} = 1\ ,$$

the maximal element of the Boolean algebra $\mathbb{B}$. $\sum_{ij}$ will always stand for $\sum_{i,j\in j^{\sigma\tau},i\neq j}$, where $\sigma\tau$ is clear from the context; similarly, $\sum_{\sigma\tau}$ means that the sum runs over all relevant $\sigma\tau$ of some fixed length ℓ (where ℓ is again clear from the context). For a relevant pair $\sigma\tau$ we let

$$A^{\sigma\tau} := \{i\in j^{\sigma\tau};\ \mu(p\cap p^{\sigma\tau}_i) < \delta\cdot\mu(p^{\sigma\tau}_i)\} \tag{6.1}$$

$$B^{\sigma\tau} := \{i\in j^{\sigma\tau};\ \mu(q\cap q^{\sigma\tau}_i) < \delta\cdot\mu(q^{\sigma\tau}_i)\}\setminus A^{\sigma\tau} \tag{6.2}$$

$$C^{\sigma\tau} := j^{\sigma\tau}\setminus(A^{\sigma\tau}\cup B^{\sigma\tau}) \tag{6.3}$$

Let $lh(\sigma) = lh(\tau) = m-1$. We say the relevant pair $\sigma\tau$ is **nice** iff

$$\sum_{i\in C^{\sigma\tau}} \mu(q^{\sigma\tau}_i)\cdot\mu(p^{\hat\sigma\hat\tau}_{\sigma(m-2)}) \leq \delta^{-2}\cdot a^{\sigma\tau}\cdot\epsilon\cdot m\cdot z_{m-1}. \tag{6.4}$$

More generally, if $lh(\sigma) = lh(\tau) = \ell$ (where $0 \leq \ell < m-1$), we say the pair $\sigma\tau$ is **nice** iff

$$\text{(6.5)} \qquad \text{(I)} \sum_{i \in C^{\sigma\tau}} \mu(q_i^{\sigma\tau}) \cdot \mu(p_{\sigma(\ell-1)}^{\hat{\sigma}\hat{\tau}}) \leq \delta^{-2} \cdot a^{\sigma\tau} \cdot \epsilon \cdot m \cdot z_{m-1};$$

$$\text{(6.6)} \qquad \text{(II)} \sum_{ij, \sigma^\frown\langle i\rangle\tau^\frown\langle j\rangle \text{nice}} a^{\sigma^\frown\langle i\rangle\tau^\frown\langle j\rangle} \geq (1 - \frac{y_\ell}{z_{\ell+1}}) \cdot \sum_{ij} a^{\sigma^\frown\langle i\rangle\tau^\frown\langle j\rangle}.$$

(Note that this is a definition by backwards recursion on ℓ.)

2.7 Claim: *For any* $\ell \quad (0 \leq \ell \leq m-1)$,

$$\text{(7.1)} \qquad \sum_{\sigma\tau \text{ nice}} a^{\sigma\tau} \geq (1 - \frac{1}{z_\ell}) \cdot a^\ell.$$

Proof: We first show that for any ℓ we have

$$\text{(7.2)} \qquad \delta^{-2} \cdot a^\ell \cdot \epsilon \cdot m \geq \sum_{\sigma\tau} \sum_{i \in C^{\sigma\tau}} \mu(q_i^{\sigma\tau}) \cdot \mu(p_{\sigma(\ell-1)}^{\hat{\sigma}\hat{\tau}}).$$

By construction (5.4), $\sum_{T \cap Z^s \neq \emptyset} a_T \geq m \cdot (1-\epsilon)$; i.e.

$$\sum_{T \cap Z^s \neq \emptyset} \sum_{j \in m} \frac{a_T^j}{a^j - a_s^j} \geq m \cdot (1 - \epsilon).$$

Hence $\sum_{T \cap Z^s \neq \emptyset} a_T^\ell \geq (a^\ell - a_s^\ell)(1 - \epsilon \cdot m)$. As $(p,q) \Vdash$"$s \in \check{T} \wedge \check{T}_s \restriction n \cap Z^s = \emptyset$" we get (using (5.3) and also the definition of $C^{\sigma\tau}$ ((6.1) — (6.3)))

$$\begin{aligned} a^\ell \cdot \epsilon \cdot m \geq (a^\ell - a_s^\ell) \cdot \epsilon \cdot m \geq & \sum_{\sigma\tau} \sum_{i \in C^{\sigma\tau}} \mu(p_{\sigma(\ell-1)}^{\hat{\sigma}\hat{\tau}}) \cdot \frac{\mu(p \cap p_i^{\sigma\tau}, q \cap q_i^{\sigma\tau})}{\mu(p_i^{\sigma\tau})} \\ \geq & \delta^2 \cdot \sum_{\sigma\tau} \sum_{i \in C^{\sigma\tau}} \mu(q_i^{\sigma\tau}) \cdot \mu(p_{\sigma(\ell-1)}^{\hat{\sigma}\hat{\tau}}). \end{aligned}$$

This shows that formula (7.2) holds.

Next, we prove the claim by backwards induction. So assume $\ell = m-1$. In that case, it follows immediately from formula (7.2) and the definition of niceness (6.4) that

$$\sum_{\sigma\tau \text{ nice}} a^{\sigma\tau} \geq (1 - \frac{1}{z_{m-1}}) \cdot a^{m-1}.$$

So let $\ell < m-1$ and assume the claim has been proved for $\ell+1$. We let $\Sigma(I) := \{\sigma\tau;\ lh(\sigma) = lh(\tau) = \ell$ and $\sigma\tau$ satisfies (I) of the definition of niceness$\}$ and $\Sigma(II) := \{\sigma\tau;\ lh(\sigma) = lh(\tau) = \ell$ and $\sigma\tau$ satisfies (II) of the definition of niceness$\}$. By the argument of the preceding paragraph we know that

$$\sum_{\sigma\tau\in\Sigma(I)} a^{\sigma\tau} \geq (1 - \frac{1}{z_{m-1}}) \cdot a^{\ell}.$$

We claim that

$$\sum_{\sigma\tau\in\Sigma(II)} \sum_{ij} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} \geq (1 - \frac{1}{y_\ell}) \cdot a^{\ell+1}. \tag{7.3}$$

For suppose not. Then

$$\sum_{\sigma\tau\notin\Sigma(II)} \sum_{ij} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} > \frac{1}{y_\ell} \cdot a^{\ell+1}.$$

But if $\sigma\tau$ does not satisfy (II) then

$$\sum_{ij,\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle \text{not nice}} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} > \frac{y_\ell}{z_{\ell+1}} \cdot \sum_{ij} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle}.$$

Hence

$$\sum_{\sigma\tau\notin\Sigma(II)} \sum_{ij,\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle \text{not nice}} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} > \frac{1}{z_{\ell+1}} \cdot a^{\ell+1},$$

contradicting the induction hypothesis.

As

$$(1-\epsilon)\cdot a^{\ell} \leq \frac{1}{1-\epsilon} \cdot a^{\ell+1} + \sum_{\sigma\tau} \sum_{i\in j^{\sigma\tau}} \mu(p_i^{\sigma\tau}, q_i^{\sigma\tau}) < \frac{1}{1-\epsilon} \cdot a^{\ell+1} + \frac{1}{v} \cdot \frac{1}{1-\epsilon} \cdot a^{\ell}$$

((5.1) and (5.2)), we have

$$\frac{1}{(1-\epsilon)^2} \cdot a^{\ell+1} \geq (1 - \frac{1}{v} \cdot \frac{1}{(1-\epsilon)^2}) \cdot a^{\ell}.$$

Hence by (7.3)

$$\begin{aligned}\sum_{\sigma\tau\in\Sigma(II)} a^{\sigma\tau} &\geq \sum_{\sigma\tau\in\Sigma(II)} \sum_{ij} a^{\sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle} \\ &\geq (1 - \frac{1}{y_\ell}) \cdot a^{\ell+1} \geq (1 - \frac{1}{y_\ell} - \frac{1}{v} \cdot \frac{1}{(1-\epsilon)^2} - 2\cdot\epsilon) \cdot a^{\ell}.\end{aligned}$$

Putting everything together we get that

$$\sum_{\sigma\tau \text{ nice}} a^{\sigma\tau} \geq (1 - \frac{1}{z_{m-1}} - \frac{1}{y_\ell} - \frac{1}{v} \cdot \frac{1}{(1-\epsilon)^2} - 2\cdot\epsilon) \cdot a^{\ell} \geq (1 - \frac{1}{z_\ell}) \cdot a^{\ell}. \quad \blacksquare$$

This shows in particular that the pair $\langle\rangle\langle\rangle$ is nice.

2.8 CLAIM: *If $\sigma\tau$ is nice of length ℓ $(0 \le \ell \le m-1)$ then*

$$\mu(p \cap p^{\hat\sigma\hat\tau}_{\sigma(\ell-1)}, q \cap q^{\hat\sigma\hat\tau}_{\tau(\ell-1)}) < (\frac{1}{4^{m-\ell}} + \zeta_\ell) \cdot \mu(p^{\hat\sigma\hat\tau}_{\sigma(\ell-1)}, q^{\hat\sigma\hat\tau}_{\tau(\ell-1)}), \tag{8.1}$$

where

$$\zeta_\ell := 2 \cdot (m-\ell) \cdot (\epsilon + \delta^{-2} \cdot \epsilon \cdot m \cdot z_{m-1} + \delta) + \sum_{j=\ell}^{m-2} \frac{y_j}{z_{j+1}}$$

(in particular $\zeta_0 = \zeta$).

Proof: We know that for arbitrary nice $\sigma\tau$,

$$\delta^{-2} \cdot a^{\sigma\tau} \cdot \epsilon \cdot m \cdot z_{m-1} \ge \sum_{i \in C^{\sigma\tau}} \mu(q_i^{\sigma\tau}) \cdot \mu(p^{\hat\sigma\hat\tau}_{\sigma(\ell-1)})$$

(cf (6.4) and (6.5)), and, by symmetry (because our conditions are relatively quadratic),

$$\delta^{-2} \cdot a^{\sigma\tau} \cdot \epsilon \cdot m \cdot z_{m-1} \ge \sum_{i \in C^{\sigma\tau}} \mu(p_i^{\sigma\tau}) \cdot \mu(q^{\hat\sigma\hat\tau}_{\tau(\ell-1)}).$$

Also

$$\delta \cdot \sum_{i \in j^{\sigma\tau}} \mu(p_i^{\sigma\tau}) > \sum_{i \in A^{\sigma\tau}} \mu(p_i^{\sigma\tau} \cap p) \quad \text{and} \quad \delta \cdot \sum_{i \in j^{\sigma\tau}} \mu(q_i^{\sigma\tau}) > \sum_{i \in B^{\sigma\tau}} \mu(q_i^{\sigma\tau} \cap q).$$

So it suffices to calculate $\sum_{i \in B^{\sigma\tau}, j \in A^{\sigma\tau}} \mu(p_i^{\sigma\tau} \cap p, q_j^{\sigma\tau} \cap q)$.

For this, we make again backwards induction on ℓ. Assume $\ell = m-1$. Then the disjointness of $B^{\sigma\tau}$ and $A^{\sigma\tau}$ (see (6.1) - (6.3)) implies that

$$\sum_{i \in B^{\sigma\tau}, j \in A^{\sigma\tau}} \mu(p \cap p_i^{\sigma\tau}, q \cap q_j^{\sigma\tau}) < \frac{1}{4} \cdot \mu(p^{\hat\sigma\hat\tau}_{\sigma(m-2)}, q^{\hat\sigma\hat\tau}_{\tau(m-2)}).$$

Now it follows from the discussion in the preceding paragraph (and (5.2)) that formula (8.1) holds for $\ell = m-1$.

So assume the claim has been proved for $\ell + 1 \le m-1$. Let $\sigma\tau$ be nice of length ℓ. Then (6.6)

$$\sum_{ij, \sigma^\frown\langle i\rangle \tau^\frown\langle j\rangle \text{not nice}} a^{\sigma^\frown\langle i\rangle \tau^\frown\langle j\rangle} \le \frac{y_\ell}{z_{\ell+1}} \cdot \mu(p^{\hat\sigma\hat\tau}_{\sigma(\ell-1)}, q^{\hat\sigma\hat\tau}_{\tau(\ell-1)}). \tag{8.2}$$

And by induction (and the disjointness of $B^{\sigma\tau}$ and $A^{\sigma\tau}$) we have

$$\sum_{\substack{i\in B^{\sigma\tau}\\ j\in A^{\sigma\tau}\\ \sigma\hat{}\langle i\rangle\tau\hat{}\langle j\rangle\text{nice}}} \mu(p\cap p_i^{\sigma\tau}, q\cap q_i^{\sigma\tau}) < (\frac{1}{4^{m-\ell-1}}+\zeta_{\ell+1})\cdot \sum_{\substack{i\in B^{\sigma\tau}\\ j\in A^{\sigma\tau}}} \mu(p_i^{\sigma\tau}, q_i^{\sigma\tau}) \tag{8.3}$$

$$< \frac{1}{4}\cdot(\frac{1}{4^{m-\ell-1}}+\zeta_{\ell+1})\cdot\mu(p^{\hat{\sigma}\hat{\tau}}_{\sigma(\ell-1)}, q^{\hat{\sigma}\hat{\tau}}_{\tau(\ell-1)}).$$

Putting everything ((5.2), the first paragraph of this proof, (8.2), (8.3)) together we get again that formula (8.1) holds. ∎

The main claim 2.6 now follows from claims 2.7 and 2.8. ∎

2.9 Proof of Theorem 3 from the Main Claim 2.6: As remarked in 2.5 we let $N := \{x \in 2^\omega;\ \exists^\infty m\,(x\restriction n_m \in Z_m)\}$. Then

$$\sum_m \frac{|Z_m|}{2^{n_m}} \le \sum_m 2^{-m} < \infty.$$

Hence N is a null set coded in V. We claim that

$$\Vdash_{\mathbb{B}\times\mathbb{B}} \check{T}\cap N \ne \emptyset.$$

We first note that it suffices to prove

$$\Vdash_{\mathbb{B}\times\mathbb{B}} \forall \ell\in\omega\ \forall s\in\check{T}\ (lh(s)=n_\ell \Rightarrow \exists^\infty m\ge\ell\ \exists t\ (lh(t)=n_m \wedge s\subseteq t\wedge t\in\check{T}\cap Z_m)).$$

For if the latter holds then we can recursively construct an $x\in\check{T}[G]\cap N$ in the generic extension $V[G]$.

So assume that there is a $(p,q)\in\mathbb{B}\times\mathbb{B}$, an $\ell\in\omega$ and an s of length n_ℓ such that

$$(p,q)\Vdash s\in\check{T}\wedge\forall m>\ell\ \forall t\ ((lh(t)=n_m\wedge s\subseteq t)\Rightarrow t\notin\check{T}\cap Z_m);$$

i.e.

$$(p,q)\Vdash s\in\check{T}\wedge\forall m>\ell\ (Z_m^s\cap\check{T}=\emptyset).$$

Without loss (p,q) is quadratic. Choose $m\ge\ell$ so large that $\mu(p,q)\ge 1/4^m+\zeta_m$. Then

$$(p,q)\Vdash s\in\check{T}\wedge Z_m^s\cap\check{T}=\emptyset$$

contradicts the main claim 2.6. ∎

3. Final remarks

3.1 We discuss the relationship between $\mathbb{C} * \mathbb{B}$, $\mathbb{B} * \mathbb{C} \cong \mathbb{B} \times \mathbb{C}$ and $\mathbb{B} \times \mathbb{B}$. Truss [T 2] proved that $\mathbb{C} * \mathbb{B}$ cannot be completely embedded in $\mathbb{B} * \mathbb{C}$ by showing that the former adds a new uncountable subset of ω_1 containing no old countable subset whereas the latter does not. In fact, he proved [T 2, Theorem 3.1] that any uncountable subset of ω_1 in $V[r]$, where r is random over V, contains a countable subset in V; the rest follows from the fact that $\mathbb{C}$ has a countable dense subset. It is easy to see that Truss' argument for $\mathbb{B}$ can be generalized to $\mathbb{B} \times \mathbb{B}$ so that $\mathbb{C} * \mathbb{B}$ cannot be completely embedded in $\mathbb{B} \times \mathbb{B}$ either.

Another argument for showing that $\mathbb{C} * \mathbb{B}$ cannot be completely embedded in $\mathbb{B} * \mathbb{C}$ is by remarking that the former produces two random reals the sum of which is Cohen (namely, let c be Cohen over V and r random over $V[c]$; then both r and $c - r$ are random over V) whereas the latter does not (by [JS 2, 2.3], if we force with $\mathbb{C}$ over $V[r]$, r random over V, then no new real is random over V; so the sum of two random reals must lie in $V[r]$ and cannot be Cohen). $\mathbb{B} \times \mathbb{B}$ also produces two random reals the sum of which is Cohen (by [Je 2, part I, 5.9], if r_0, r_1 are the random reals added by $\mathbb{B} \times \mathbb{B}$, then $r_0 + r_1$ is Cohen). So $\mathbb{B} \times \mathbb{B}$ cannot be completely embedded in $\mathbb{B} * \mathbb{C}$.

On the other hand, Pawlikowski (see the last paragraph of §3 in [Pa]) proved that $\mathbb{B} * \mathbb{C}$ can be completely embedded into any algebra adding both Cohen and random reals; in particular $\mathbb{B} * \mathbb{C} <_c \mathbb{C} * \mathbb{B}$ and $\mathbb{B} * \mathbb{C} <_c \mathbb{B} \times \mathbb{B}$, where $<_c$ means **is complete subalgebra of**. Hence the only question left open is whether $\mathbb{B} \times \mathbb{B} <_c \mathbb{C} * \mathbb{B}$ (Question 1).

3.2 We continue with a remark concerning Question 2. Namely, suppose there are models of ZFC $M \subseteq N$ such that N contains both a dominating and a random real over M, but does not contain a perfect set of random reals over M. Without loss, $N = M[r][d]$, where r is random over M, and d is dominating over M. By the ω^ω-bounding property of random forcing [Je 2, part I, 3.3 (a)], d is dominating over $M[r]$. Let c be Cohen over N. A result of Truss [T 1, Lemma 6.1] says that $d + c$ is Hechler over $M[r]$.

(Recall that Hechler forcing $\mathbb{D}$ is defined as follows:

$$\mathbb{D} := \{(n, f);\ n \in \omega\ \wedge\ f \in \omega^\omega\},$$

$(n, g) \leq (m, f)$ iff $n \geq m$ and $\forall \ell \in \omega\ (g(\ell) \geq f(\ell))$ and $f \restriction m = g \restriction m$. $\mathbb{D}$ generically adds a dominating real.)

By [JS 2, 2.3] there is no new real random over M in $N[c]$, in particular, there is no perfect set of random reals over M in $N[c]$, thus showing that $\mathbb{B}*\mathbb{D} \cong \mathbb{B}\times\mathbb{D}$ does not add a perfect set of random reals. Hence Question 2 is equivalent to

QUESTION 2': *Does $\mathbb{B}\times\mathbb{D}$ add a perfect set of random reals?*

We will now see that for many forcing notions $\mathbb{P}$ adding a dominating real it is true that $\mathbb{B}\times\mathbb{P}$ adds a perfect set of random reals.

3.3 PROPOSITION: *Let $\mathbb{M}$ be Mathias forcing. Then $\mathbb{B}\times\mathbb{M}$ adds a perfect set of random reals.*

Remark: Mathias forcing is defined as follows.

$$\mathbb{M} := \{(s,S);\ s\in\omega^{<\omega}\ \wedge\ S\in[\omega]^\omega\ \wedge\ \max s<\min S\},$$

$(t,T)\le(s,S)$ iff $t\supseteq s$ and $T\subseteq S$ and $\forall n\in \mathrm{dom}(t)\setminus\mathrm{dom}(s)\ (t(n)\in S)$.

Sketch of proof: In $V[G]$, where G is $\mathbb{B}\times\mathbb{M}$-generic over V, let r be the random real and d the Mathias real (which is dominating). We claim that

$$\begin{aligned}T := \{f\in 2^\omega;\ \forall n\in\omega\ (f{\restriction}[d(n),d(n+1))\\ = r{\restriction}[d(n),d(n+1))\ \vee\ f{\restriction}[d(n),d(n+1))\\ = (1-r){\restriction}[d(n),d(n+1)))\}\end{aligned}$$

is a perfect set of reals random over V in $V[G]$.

We show that given a null set $N\in V$ and a condition $(B,(s,S))\in\mathbb{B}\times\mathbb{M}$, there is a $(B',(s,S'))\le(B,(s,S))$ such that

$$(B',(s,S'))\Vdash \check{T}\cap N=\emptyset,$$

where $\check{T}$ is a name for the perfect set defined above. First note that by [Ba] there are partitions $\{I_i;\ i\in\omega\}$ and $\{I'_i;\ i\in\omega\}$ of ω into finite intervals with $\max(I_i)<\min(I_j)$, $\max(I'_i)<\min(I'_j)$ for $i<j$, sequences $\langle J_i;\ i\in\omega\rangle$ and $\langle J'_i;\ i\in\omega\rangle$ such that

$$J_i\subseteq 2^{I_i},\quad J'_i\subseteq 2^{I'_i},\quad \sum_{i\in\omega}\frac{|J_i|}{2^{|I_i|}}<\infty,\quad \sum_{i\in\omega}\frac{|J'_i|}{2^{|I'_i|}}<\infty$$

and

$$N\subseteq\{f\in 2^\omega;\ \exists^\infty n\ (f{\restriction}I_n\in J_n)\}\cup\{f\in 2^\omega;\ \exists^\infty n\ (f{\restriction}I'_n\in J'_n)\}.$$

Now find $S' \subseteq S$ such that for all $n \in \omega$, $|S' \cap I_n| \leq 1$ and $|S' \cap I'_n| \leq 1$. Let i_n be the unique element of $S' \cap I_n$, and i'_n the unique element of $S' \cap I'_n$ (if it exists — if not, let $i_n \in I_n$ and $i'_n \in I'_n$ be arbitrary). Set

$$K_n := \{s \in 2^{I_n};\ s \in J_n \ \vee\ 1 - s \in J_n \ \vee (s \restriction i_n) \cup (1 - s \restriction [i_n, \infty)) \in J_n \\ \vee\ (1 - s \restriction i_n) \cup (s \restriction [i_n, \infty)) \in J_n\};$$

similarly we define K'_n. We choose $B' \leq B$ such that

$$B' \cap (\{f \in 2^\omega;\ \exists^\infty n\ (f \restriction I_n \in K_n)\} \cup \{f \in 2^\omega;\ \exists^\infty n\ (f \restriction I'_n \in K'_n)\}) = \emptyset.$$

We leave it to the reader to verify that this works. ∎

We note that a similar argument works for Laver forcing, for Mathias forcing with a q-point ultrafilter etc. But it is unclear whether $\mathbb{B} * \mathbb{M}$ adds a perfect set of random reals.

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